ECE 558 Project 2 Solutions

① Application of conditional probability (Rule 7): occurrence of an event which is dpt. on a number of mutually exclusive events.

Box 1	Box 2	Box 3
3R	1R	3R
3W	1W	1W
2B	1B	5B

Box 1 $\xrightarrow{\text{draw one}}$ Box 2 $\xrightarrow{\text{draw one}}$ Box 3

Pr(# R in Box3 < 3) = 0
Pr(# R in Box3 < 4) = ?
Pr(# R in Box3 < 5) = 1

Let R_i be event that Red ball is drawn from Box i

W_i —— " —— white —— " ——

B_i —— " —— Blue —— " ——

a) $Pr(R_2) = Pr(R_2 \cap R_1) + Pr(R_2 \cap \overline{R_1})$

$= Pr(R_2 | R_1) \cdot Pr(R_1) + Pr(R_2 | \overline{R_1}) \cdot Pr(\overline{R_1})$

$= \frac{2}{4} \times \frac{3}{8} + \frac{1}{4} \times (1 - \frac{3}{8})$

$= 11/32$

$Pr(W_2) = Pr(W_2 | W_1) \cdot Pr(W_1) + Pr(W_2 | \overline{W_1}) \cdot Pr(\overline{W_1})$

$= \frac{2}{4} \times \frac{3}{8} + \frac{1}{4} \times (1 - \frac{3}{8})$

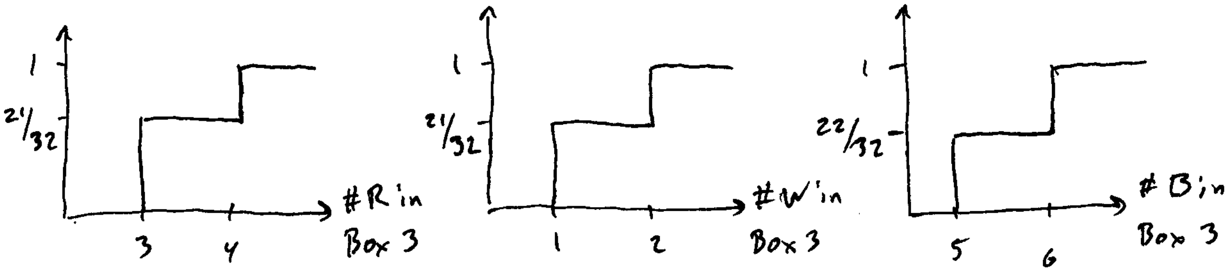
$= 11/32$

$Pr(B_2) = Pr(B_2 | B_1) \cdot Pr(B_1) + Pr(B_2 | \overline{B_1}) \cdot Pr(\overline{B_1})$

$= \frac{2}{4} \times \frac{2}{8} + \frac{1}{4} \times (1 - \frac{2}{8})$

$= 10/32$

cumulative distr. function



b) $E(\#R \text{ in Box } 3) = 3 \times \frac{21}{32} + 4 \times \frac{11}{32} = 107/32$

$E(\#W \text{ in Box} 3) = 1 \times \frac{21}{32} + 2 \times \frac{11}{32} = 43/32$

$E(\#B \text{ in Box } 3) = 5 \times \frac{22}{32} + 6 \times \frac{10}{32} = 170/32$

c) $V(x) = E(x^2) - E^2(x)$

$V(\#R \text{ in Box} 3) = (3)^2 \times \frac{21}{32} + (4)^2 \times \frac{11}{32} - (107/32)^2 = 0.2256$

$V(\#W \text{ in Box} 3) = (1)^2 \times \frac{21}{32} + (2)^2 \times \frac{11}{32} - (43/32)^2 = 0.2256$

$V(\#B \text{ in Box} 3) = (5)^2 \times \frac{22}{32} + (6)^2 \times \frac{10}{32} - (170/32)^2 = 0.2148$

d) $Pr(W_1|B_2) = \frac{Pr(W_1 \cap B_2)}{Pr(B_2)} = \frac{Pr(B_2 \cap W_1)}{Pr(B_2)} = \frac{Pr(B_2|W_1) \cdot Pr(W_1)}{Pr(B_2)}$

$= \frac{\frac{1}{4} \times \frac{3}{8}}{10/32} = 3/10$ Note: $Pr(W_1) = 3/8$

② Two fair coins $p = 0.5$ (Heads)

$(p+q)^2 = p^2 + 2pq + q^2$ (3 outcomes)

$\uparrow$ two H, $\uparrow$ One H, $\uparrow$ no H

x_i: \$2, \$1, −\$5 (payoff)

$E(x) = \sum_{i=1}^{3} x_i p_i = 2 \cdot (\frac{1}{2})^2 + 1 \cdot 2 \cdot (\frac{1}{2})(\frac{1}{2}) - 5 \cdot (\frac{1}{2})^2 = -0.25$ No, should not play.

③ $p = 0.8$ (Prob. of good casting), $q = 0.2$

Want Prob. of exactly 2 bad castings (Recall: $(p+q)^8$)

$_8C_6 \, p^6 q^2 = \frac{8!}{6!\,(8-6)!} (0.8)^6 (0.2)^2 = 28 \cdot (0.8)^6 (0.2)^2 = 0.2936$

④ Prob of pair of dice summing to 8: $\{(2,6), (3,5), (4,4), (5,3), (6,2)\} \Rightarrow 5/36$

$p = 5/36$

a) $(p+q)^4$: Want exactly 3 successes out of 4

$${}_4C_3\, p^3 q^1 = \frac{4!}{3!\,(4-3)!}\left(\tfrac{5}{36}\right)^3\left(1-\tfrac{5}{36}\right) = 0.009228$$

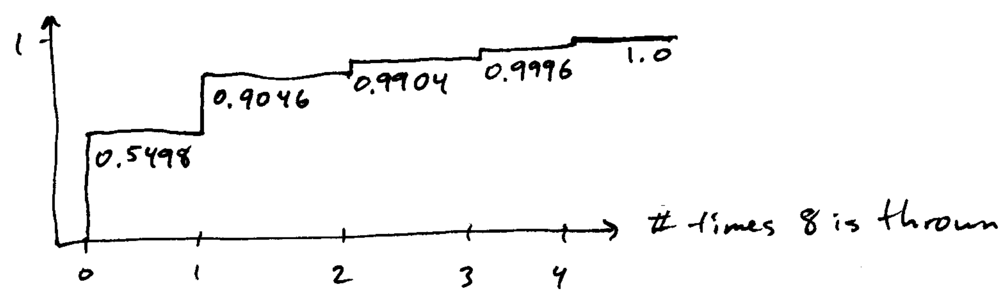
b) want at least 2 successes out of 4 trials

$${}_4C_2\, p^2 q^2 + {}_4C_3\, p^3 q^1 + {}_4C_4\, p^4$$

$$= \frac{4!}{2!\,(4-2)!}\left(\tfrac{5}{36}\right)^2\left(1-\tfrac{5}{36}\right)^2 + 0.009228 + \frac{4!}{4!\,(4-4)!}\left(\tfrac{5}{36}\right)^4$$

$$= 0.08582 + \cdots + \cdots = 0.09542$$

c)



Note: ${}_4C_0\, q^4 = 0.5498$

$${}_4C_1\, p^1 q^3 = \frac{4!}{1!\,(4-1)!}\left(\tfrac{5}{36}\right)\left(1-\tfrac{5}{36}\right)^3 = 0.3547$$

(5) 10 lines, $q = 0.8$ (Prob. that any individual line is busy)

a) Pr(at least 3 lines free) $= 1 - Pr$(less than 3 are free)

$$= 1 - \left[{}_{10}C_0\, q^{10} + {}_{10}C_1\, p^1 q^9 + {}_{10}C_2\, p^2 q^8\right]$$

$$= 1 - \{0.1074 + 0.2684 + 0.3020\}$$

$$= 0.3222$$

b) $$E(\#\text{ free lines}) = \sum_{i=0}^{10} i \cdot p_i = 0 + 1\cdot {}_{10}C_1\, p^1 q^9 + 2\cdot {}_{10}C_2\, p^2 q^8 + \cdots + 10\cdot {}_{10}C_{10}\, p^{10}$$

$= 2$ lines

(see Matlab code)

```
%
% telephone line problem
%
% we know that the binomial distribution leads to an
% expected value of np (derived in lecture), but this
% code goes through the brute force sum of x_i*p_i
% for your educational pleasure
%
n=10;  % # of independent lines
p=0.2; % probability that any individual line is free
exp_val=0;
for i=0:n,
    p_i=nchoosek(n,i)*p^i*(1-p)^(n-i)
    exp_val=exp_val + i*p_i;
end
fprintf('exp_val = %5.3f, n = %2d, p = %5.3f\n', exp_val, n, p);
%
% MATLAB output
%
p_i =
    0.1074
p_i =
    0.2684
p_i =
    0.3020
p_i =
    0.2013
p_i =
    0.0881
p_i =
    0.0264
p_i =
    0.0055
p_i =
  7.8643e-004
p_i =
  7.3728e-005
p_i =
  4.0960e-006
p_i =
  1.0240e-007
%
% expected value (as expected, it's 2)
%
exp_val = 2.000, n = 10, p = 0.200
```