This is an open-book, open-notes exam. The use of electronic calculators is permitted. The
exam lasts 75 minutes.

There are four questions on the exam. The first asks two “short answer” questions. The
remaining three are problems to work. Do all your work on these pages and indicate your
final answer clearly. For each of the problems I’ve provided an extra worksheet. Use the
backs of the sheets if necessary. Neatness and clarity are important and can influence your
grade!

Each problem is weighted toward the final total as shown below.

**Grades**

1. ________________ (15 pts.)
2. ________________ (25 pts.)
3. ________________ (30 pts.)
4. ________________ (30 pts.)

Total ________________ (100 pts.)
1. **[5 points each, total 15]** Please provide a short answer for parts (a), (b) and (c).

   (a) Frequently a likelihood ratio test based on a measurement of a continuous random observation $x$ will take the form

   \[
   \frac{d_1}{d_0} \begin{array}{c} x \\ \gamma \end{array}
   \]

   for some threshold $\gamma$. Explain why in this case you cannot change the threshold $\gamma$ to increase $P_D$ while simultaneously decreasing $P_{FA}$.

   (b) Under what conditions is the MAP (maximum a posteriori) detector equal to the ML (maximum likelihood) detector?

   (c) If we use a MAP detector, are we adopting a Bayesian approach, a classical approach, or either approach to detection? Answer the same question but in the case of the ML detector.
2. **[25 points]** We make three successive observations

\[ x(n) = s(n) + w(n), \quad n = 0, 1, 2 \]

with \( w(n) \) iid with pdf

\[ p(w) = \frac{1}{2} e^{-|w|}. \]

The signal \( s(n) \) is 0 for \( n = 0, 1, 2 \) under \( \mathcal{H}_0 \) (no signal present), and under \( \mathcal{H}_1 \) we have \( s(0) = s(2) = A/2 \) and \( s(1) = A \) for a known value \( A > 0 \) (signal present).

Develop a test in the form of a comparison of a statistic to a threshold in order to detect from the observations whether the signal is present. You want your test to have the maximum probability of choosing \( \mathcal{H}_1 \) if the signal is present, among all tests that have the same probability of choosing \( \mathcal{H}_1 \) when no signal is present.
EXTRA WORKSHEET for problem 2
3. **[30 points]** A smoke detector has a sensor that produces an output \( x(n) \) at each time sample \( n \) that takes one of four values: 0, 1, 2, and 3. With \( \mathcal{H}_0 \) the hypothesis that there is no fire (no smoke), and \( \mathcal{H}_1 \) the hypothesis that this is a fire (smoke present), the probability of these output values is

\[
P(x|\mathcal{H}_0) = \begin{cases} 
3/4, & x = 0 \\
1/8, & x = 1 \\
1/16, & x = 2 \\
1/16, & x = 3 
\end{cases} \quad P(x|\mathcal{H}_1) = \begin{cases} 
1/8, & x = 0 \\
3/16, & x = 1 \\
1/2, & x = 2 \\
3/16, & x = 3 
\end{cases}
\]

with \( \{x(n)\} \) iid.

(a) Determine how to use a single observation \( x(n) \) to trigger or not to trigger the smoke alarm if

\[
P(\mathcal{H}_0) = 1000 \ P(\mathcal{H}_1)
\]

and it is considered 10,000 times more costly to fail to trigger the smoke alarm in the event of a fire than it is to have a false alarm. You detector should minimize the expected perceived cost. What is the false alarm probability at each time \( n \)?

(b) Redesign your detector but now using \( x(n) \) and \( x(n-1) \), again to minimize the expected perceived cost. What is the false alarm rate?
EXTRA WORKSHEET for problem 3
4. [30 points] You measure iid $x(n)$, $n = 0, 1, \ldots, N-1$. Under $\mathcal{H}_0$ we have $x \sim \mathcal{N}(0, \sigma^2)$ and under $\mathcal{H}_1$ we have $x \sim \mathcal{N}(0, A\sigma^2)$ for some value of $A$.

(a) Assume that $A > 0$, but that $A$ is unknown. Show whether or not there exists a UMP (uniformly most powerful) test.

(b) Let $\hat{A}$ denote the ML (maximum likelihood) estimate of $A$ based on

$$ \{x(0), \ldots, x(N-1)\}.$$

Find the GLRT (generalized likelihood ratio test) based on a ML detector criterion in terms of this $\hat{A}$. 
EXTRA WORKSHEET for problem 4