This is an open-book, open-notes exam. The use of electronic calculators is permitted. The exam lasts 120 minutes.

There are three questions on the exam. Do all your work on these pages and indicate your final answer clearly. Two extra worksheets are provided for each problem. You may also use the backs of the sheets if necessary (but if you do, be sure to mark clearly what problem that work goes with).

Neatness and clarity are important and can influence your grade!

Each problem is weighted toward the final total as shown below.

Grades

1. ___________________ (40 pts.)
2. ___________________ (40 pts.)
3. ___________________ (20 pts.)
Total ___________________ (100 pts.)
1. [40 points] You measure $x(n)$ for $n = 0, 1, \ldots, N - 1$ and you want to determine from this data whether $x(n)$ is a noisy signal or $x(n)$ is just noise. Your model for the two situations is

$$ x(n) = w(n), \quad \text{if } x(n) \text{ is noise} $$

$$ x(n) = s(n) + w(n), \quad \text{if } x(n) \text{ is signal plus noise} $$

with $w(n) \sim \mathcal{N}(0, \sigma^2)$ IID and $s(n) \sim \mathcal{N}(\mu, \sigma_s^2)$ IID. Also, \{w(n)\} and \{s(n)\} are independent, and you know the values of $\mu$, $\sigma_s^2$, and $\sigma^2$.

The following rule is proposed. If

$$ \sum_{n=0}^{N-1} \left( x(n) + \mu \frac{\sigma^2}{\sigma_s^2} \right)^2 > \gamma, $$

then one considers \{x(n)\} to be a signal plus noise; otherwise, you conclude \{x(n)\} is just noise. For any particular value of $\gamma$, this rule has associated with it a probability of correctly determining the presence of a signal, as well as a probability of mistakenly determining the presence of a signal when \{x(n)\} is just noise.

Show that this rule maximizes the probability of correctly determining the signal’s presence among all rules that have the same probability of mistakenly determining a signal’s presence.
EXTRA WORKSHEET for problem 1
EXTRA WORKSHEET for problem 1
2. [40 points] You measure

\[ x(n) = \alpha s(n) + w(n) \]

where

\[ s(n) = \sqrt{2} \cos \left( \frac{\pi}{2} n + \frac{\pi}{4} \right) \]

while \( \alpha \) is unknown. You know \( w(n) \sim \mathcal{N}(0, \sigma^2) \) IID, but \( \sigma^2 \) is unknown.

Estimate \( \alpha \) and \( \sigma^2 \) if

\[
\begin{bmatrix}
  x(0) \\
  x(1) \\
  x(2) \\
  x(3)
\end{bmatrix} =
\begin{bmatrix}
  2 \\
  -2 \\
  -3 \\
  1
\end{bmatrix}.
\]

Explain why you think your estimates are good ones. (You don’t have to demonstrate optimality in any sense, but of course doing so provides good support for your estimate.)
EXTRA WORKSHEET for problem 2
EXTRA WORKSHEET for problem 2
3. **[20 points]** You make a (single) measurement $x$, and you know that the dependence of $x$ on an unknown parameter $\theta$, coupled with your understanding of the distribution of $\theta$ when you view it as a random variable, is such that

$$p(\theta|x) = \begin{cases} \exp[-(\theta - x)], & \theta > x \\ 0, & \theta < x \end{cases}$$

(a) Determine an estimate $\hat{\theta}$, dependent on $x$, that minimizes $E[(\theta - \hat{\theta})^2]$.

(b) Determine an estimate $\hat{\theta}$, dependent on $x$, that minimizes

$$\lim_{\delta \to 0} E[C_\delta(\theta - \hat{\theta})]$$

where

$$C_\delta(\epsilon) = \begin{cases} 1, & |\epsilon| > \delta \\ 0, & |\epsilon| < \delta \end{cases}$$
EXTRA WORKSHEET for problem 3
EXTRA WORKSHEET for problem 3