Name: ________________________________

This is an open-book, open-notes exam. The use of electronic calculators is permitted. The exam lasts 75 minutes.

There are four questions on the exam. The first is a “short answer” type question, while the remaining three are problems to work. Do all your work on these pages and indicate your final answer clearly. For each of these problems I’ve provided an extra worksheet. Use the backs of the sheets if necessary. Neatness and clarity are important and can influence your grade!

Each problem is weighted toward the final total as shown below.

Grades

1. ____________________________ (10 pts.)
2. ____________________________ (30 pts.)
3. ____________________________ (30 pts.)
4. ____________________________ (30 pts.)
Total ________________________ (100 pts.)
1. [10 points] In the linear model, we have

\[ \bar{x} = H\theta + w \]

with \( w \sim \mathcal{N}(0, C) \). We developed in class the MVUE \( \hat{\theta} = g(\bar{x}) \) in this case. Later, when we developed the BLUE, we again assumed

\[ \bar{x} = H\theta + w, \]

and the form of the BLUE was once more \( \hat{\theta} = g(\bar{x}) \). (That is, the BLUE is given by the same function of the data \( \bar{x} \) as for the MVUE in the linear model.)

However, we noted that the BLUE does not in general achieve the minimum variance possible among all estimators. Explain how this can be.
2. [30 points total] You measure IID $x(n)$ for $n = 0, \ldots, N - 1$ with
\[
p(x(n); \mu) = \begin{cases} \\
\frac{1}{\mu} e^{-x/\mu}, & x > 0 \\
0, & x < 0 \\
\end{cases}
\]
and you want to form an unbiased estimate $\hat{\mu}$ of the mean value of $x(n)$, which is the unknown parameter $\mu$.

(a) [15 points] Find the smallest value of $N$ so that it is possible to achieve
\[
\frac{\text{VAR}[\hat{\mu}]}{\mu^2} < \sqrt{\frac{1}{10}}.
\]
(That is, find $N$ so that the estimate $\hat{\mu}$ is typically with 10% of $\mu$.)

(b) [15 points] Find an unbiased estimate $\hat{\mu}$ of $\mu$ that achieves (1).
EXTRA WORKSHEET for problem 2
3. **[30 points total]** Suppose $x(0), x(1), \ldots, x(N - 1)$ are IID with each $x(n)$ distributed according to a Gamma distribution with parameters $\alpha$ and $\beta$:

$$p(x(n); \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} (x(n))^{\alpha - 1} \exp(-\beta x(n)).$$

Note that for this distribution, $E[x(n)] = \alpha/\beta$ and $\text{var}[x(n)] = \alpha/\beta^2$.

(a) **[10 points]** If $\beta$ is known and $\alpha$ is unknown, find a minimal sufficient statistic for $\theta = \alpha$.

(b) **[10 points]** Assuming that your statistic from part (a) is complete, either find an MVUE of $\alpha$ as a function of this statistic, or argue why this is difficult to do.

(c) **[10 points]** If both $\alpha$ and $\beta$ are unknown, find a minimal sufficient statistic for $\theta = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$.
EXTRA WORKSHEET for problem 3
4. [30 points] You measure $x(n)$ for $n = 0, 1, \ldots, N - 1$ with $x(n) \sim \mathcal{N}(A, A^2)$ and IID. Find the BLUE of $A$ and compare its variance to the CRLB.
EXTRA WORKSHEET for problem 4