This is an open-book, open-notes exam. The use of electronic calculators is permitted. The exam lasts 75 minutes.

There are four questions on the exam. The first asks two “short answer” questions. The remaining three are problems to work. Do all your work on these pages and indicate your final answer clearly. For each of the problems I’ve provided an extra worksheet. Use the backs of the sheets if necessary. Neatness and clarity are important and can influence your grade!

Each problem is weighted toward the final total as shown below.

Grades

1. ________________ (10 pts.)
2. ________________ (30 pts.)
3. ________________ (30 pts.)
4. ________________ (30 pts.)
Total ________________ (100 pts.)
1. [5 points each, total 10] Please provide a short answer for parts (a) and (b).

(a) Explain why the detector

\[
\ln(\Lambda(x)) \gtrless 0
\]

is called the MLE, where \(\ln(\Lambda(x))\) is the log likelihood ratio.

(b) Explain why it is not desirable to have an ROC that falls below the line \(P_D = P_{FA}\) at any point.
2. [30 points] We make a single observation $x$ with

$$x = \begin{cases} w, & \text{under } \mathcal{H}_0 \\ 6 + w, & \text{under } \mathcal{H}_1 \end{cases}$$

with

$$p(w) = \frac{1}{2} e^{-|w|}.$$

This is an instance of a signal (6) in additive noise ($w$), and we wish to detect the signal while avoiding false alarms.

(a) Find the log likelihood ratio as a function of $x$.
(b) Determine the test that maximizes $P_D$ while $P_{FA} = (2e)^{-1}$.
(c) What is $P_D$ for your test in (b)?
EXTRA WORKSHEET for problem 2
3. **[30 points]** When a system consisting of two components \( C_0 \) and \( C_1 \) fails, we make a measurement \( x \) whose pdf’s in the event of failure of component \( C_0 \) \( (p(x \mid \mathcal{H}_0)) \) and of component \( C_1 \) \( (p(x \mid \mathcal{H}_1)) \) are

\[
\begin{align*}
\mathcal{H}_0 & \quad \mathcal{H}_1 \\
0 & \quad 2 & \quad 4 & \quad 6 & \quad 8 & \quad 10
\end{align*}
\]

![Graph of p(x | H1)](image1)

![Graph of p(x | H0)](image2)

You also know that on average, for every 7 failures of \( C_0 \) there are 4 failures of \( C_1 \). Develop a test \( d(x) \) to decide whether \( C_0 \) or \( C_1 \) has failed. Your test should minimize the probability of error.
EXTRA WORKSHEET for problem 3
4. [30 points] You measure iid $x(n)$, $n = 0, 1, \ldots, N - 1$. Under $\mathcal{H}_0$

$$p(x; \mathcal{H}_0) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

while under $\mathcal{H}_1$

$$p(x; \mathcal{H}_1) = \begin{cases} \frac{1}{\mu} e^{-(x/\mu)}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

with $\mu > 0$ unknown. Show that the GLRT $L_G(x) \overset{d_1}{\underset{d_0}{\gtrless}} \gamma$ has the form

$$\bar{x} - \ln(\bar{x}) \overset{d_1}{\underset{d_0}{\gtrless}} \gamma'$$

where $\bar{x}$ is the sample mean.
EXTRA WORKSHEET for problem 4