1. (a) We have

\[ Y(z)/U(z) = \frac{z^2 - 3z + 2}{z^3 - 2z^2 - z + 1} \]

with zeros at \( z = 2, z = 1, \) and \( z = \infty, \) and poles at \( z = 2.2470, z = -0.8019, \) and \( z = 0.5550. \)

(b) Choose

\[ K = \begin{bmatrix} 4.5251 & -1.1464 & 4.2322 \end{bmatrix} \]

(c) Choose

\[ K = \begin{bmatrix} 4.875 & -1.125 & 4.375 \end{bmatrix} \]

(d)

\[ Y(z) = \frac{z^2 - 3z + 2}{z^3 - 2z^2 - z + 1} \]

\[ \frac{R(z)}{z^3 - 1.707z^2 + 1.207z - 0.3536} = \frac{(z - 1)(z - 2)}{(z - \sqrt{2}/2)(z - (0.5 + j0.5))(z - (0.5 - j0.5))} \]

2. (a) We have from \( G(s) = 2/s^2 \) that

\[ y(t) = \int \int 2 u(\tau)d\tau \]

\[ \dot{y}(t) = \int 2 u(\tau)d\tau \]

\[ \ddot{y}(t) = 2 u(t). \]

Thus \( \dot{x}_2(t) = \ddot{y}(t) = 2 u(t) \) and \( \dot{x}_1(t) = \dot{y}(t) = x_2(t), \) yielding

\[ \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t) \]

\[ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \]

as the state variable description for this system.

(b) The discrete equivalent (ZOH equivalent) of the plant is

\[ x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(k) \]

\[ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \]

Using Ackermann’s formula we have

\[ K = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}^{-1} \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^2 - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + 0.5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \]

\[ = \begin{bmatrix} 1/4 & 3/8 \end{bmatrix} \]
3. (a) We have

\[ x(k + 1) = (A - BK)x(k) + Br(k) \]
\[ u(k) = -Kx(k) + r(k) \]

and hence

\[
\frac{U(z)}{R(z)} = \frac{\text{det} \begin{bmatrix} zI - A + BK & B \\ K & 1 \end{bmatrix}}{\text{det}(zI - A + BK)} = \frac{\text{det} \begin{bmatrix} zI - A & B \\ 0 & 1 \end{bmatrix}}{\text{det}(zI - A + BK)} = \frac{\text{det}(zI - A)}{\text{det}(zI - A + BK)}
\]

(b) Yes. \( U/R \) and \( Y/R \) share the same poles (which are the roots of \( \text{det}(zI - A + BK) \)).

(c) \( U(z)/R(z) \) is minimum phase only if the plant is stable, because the zeros of \( U(z)/R(z) \) are the eigenvalues of \( A \) (that is, the roots of \( \text{det}(zI - A) \)), which are the poles of the plant.

(d) \( U(z)/R(z) \) is causal since

\[
\frac{U(z)}{R(z)} = -K(zI - A + BK)^{-1}B + 1,
\]

which is a proper rational function.

4. MATLAB code at the course web site provides an implementation of one solution.

(a) We use \( T = 0.04 \) (one tenth the rise time) and do a design to place the closed loop poles at \( z = 0.8 \pm 0.2 \) and at \( z = 0 \). Using MATLAB to calculate a ZOH equivalent of \( G(s) \) and convert it to state space form yields

\[
x(k + 1) = \begin{bmatrix} 2.6311 & -1.1376 & 0.6440 \\ 2.0000 & 0 & 0 \\ 0 & 0.5000 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0.0156 \\ 0 \\ 0 \end{bmatrix} u(k)
\]
\[
y(k) = \begin{bmatrix} 0.0061 & 0.0110 & 0.0049 \end{bmatrix} x(k)
\]

We compute \( K \) and get

\[
K = \begin{bmatrix} 65.9910 & -51.0447 & 41.2183 \end{bmatrix}.
\]

The plot of the resulting step response is shown in Figure 1, after normalization to a unit value of DC gain.
Figure 1: Step response for problem 4.
(b) The plant ZOH equivalent is
\[
\frac{9.5821 \times 10^{-5}(z + 3.354)(z + 0.2393)}{(z - 1)(z - 0.9608)(z - 0.6703)}.
\]

With \( u(k) = r(k) - Kx(k) \) we get a closed loop transfer function of
\[
\frac{9.5821 \times 10^{-5}(z + 3.354)(z + 0.2393)}{z(z - 0.8 - j0.2)(z - 0.8 + j0.2)}.
\]

The DC gain of this transfer function is not equal to 1, so this system can’t track ramps with finite error (as the system needs to be able to track steps with zero steady state error in order to track ramps with finite error).