(1) Let $\bar{x} = 1$. We have that $0 \leq \alpha \leq \bar{x}$ implies

$$
\begin{align*}
    f(\bar{x} + \alpha \Delta x) &= f(\bar{x} + \alpha (x^* - \bar{x})) \\
    &= f([1-\alpha] \bar{x} + \alpha x^*) \\
    &\leq [1-\alpha] f(\bar{x}) + \alpha f(x^*) \\
    &\quad \text{(by convexity)} \\
    &< [1-\alpha] f(\bar{x}) + \alpha f(\bar{x}) \\
    &\quad \text{(since $\bar{x} \neq \arg\min_{x \in \mathbb{R}^m} f(x)$)} \\
    &= f(\bar{x})
\end{align*}
$$

which is the definition of a descent direction.

(2)(i) $\forall x$, $\nabla f(x) = -\exp(-x)$

(ii) $\forall x$, $\nabla^2 f(x) = \exp(-x)$

(iii) Since $\nabla^2 f(x) > 0$ for all $x$, it is positive definite throughout $\mathbb{R}$. Then by Thm. 2.7, $f$ is convex.
(2)(iv) Since $\nabla f(x) < 0$ for all $x$, there is no $x$ satisfying $\nabla f(x) = 0$.

(v) Since $\nabla f(x^*) = 0$ is the necessary and sufficient condition for $x^*$ to be a minimizer of $f$, and since there is an $x^*$ with $\nabla f(x^*) = 0$, there is no minimizer of $f$.

(3)(i) MATLAB requires two function evaluations to obtain $x^* = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(ii) The condition number is 1.

(4)(i) MATLAB requires 8 iterations to get an approximate solution of $\begin{bmatrix} 0.9990 \\ 2.9990 \end{bmatrix}$.

(ii) The condition number is 19.
(4)(iii) MATLAB requires 54 iterations to obtain an approximate solution of
\[ \begin{bmatrix} 0.9983 \\ 2.9982 \\ 0.9983 \\ 2.9984 \end{bmatrix} \]

(iv) The condition number is approximately 21.97.

(5) We have \( \frac{\partial f}{\partial x}(x) = x^TQ + c^T \)

\[ \frac{\partial^2 f}{\partial x^2}(x) = Q \]

(i) The 1st order necessary condition for \( x^* \) to be an optimum are that
\[ \nabla f(x^*) = 0 \]

\[ \implies Qx^* + c = 0 \]
(5) (ii) For step size 1, the Newton-Raphson update for the $v^{th}$ iteration is
\[ Q \Delta x^{(v)} = -(Qx^{(v)} + c) \]
\[ x^{(v+1)} = x^{(v)} + \Delta x^{(v)} \]
Since $Q$ is non-singular,
\[ x^{(v+1)} = x^{(v)} + Q^{-1}(-Qx^{(v)} - c) \]
\[ = -Q^{-1}c \]
Therefore
\[ \nabla f(x^{(v+1)}) = Qx^{(v+1)} + c \]
\[ = -QQ^{-1}c + c \]
\[ = 0 \]
Thus, for any $x^{(0)}$, $x^{(1)}$ satisfies the 1st order necessary conditions.

(iii) $f(x^{(0)}) - f(x^{(1)}) = \left( \frac{1}{2}(x^{(0)})^TQx^{(0)} + c + x^{(0)} + d \right) - \left( \frac{1}{2}(x^{(1)})^TQx^{(1)} + c + x^{(1)} + d \right)$
\[
\begin{align*}
&= \left( \frac{1}{2} x^{(0)^T} Q x^{(0)} + c^T x^{(0)} + d \right) \\
&\quad - \left( \frac{1}{2} (x^{(0)} + \Delta x^{(0)})^T Q (x^{(0)} + \Delta x^{(0)}) \\
&\quad \quad + c^T (x^{(0)} + \Delta x^{(0)}) + d \right) \\
&= H x^{(0)^T} Q \\
&= -\frac{1}{2} \Delta x^{(0)^T} Q \Delta x^{(0)} - \Delta x^{(0)^T} Q x^{(0)} - c^T \Delta x^{(0)} \\
&= -\frac{1}{2} \Delta x^{(0)^T} Q \Delta x^{(0)} - \Delta x^{(0)^T} \left[ -Q \Delta x^{(0)} - c \right] \\
&\quad - c^T \Delta x^{(0)} \\
&\quad \text{using the result from part (ii)} \\
&= \frac{1}{2} \Delta x^{(0)^T} Q \Delta x^{(0)} \\
\end{align*}
\]

(iv) Yes, if \( Q \) is not positive semidefinite.