Go to the website
and follow the link on the left side of the page under “Resources and solutions” (click
on “View material...” to follow the link) to view the online support material for the
course textbook. Download the appendices and read Appendix A.

- Read Chapter 1 of the text.
- Read Chapter 2 of the text.

Work the following problems and hand in your solutions to them.

1. Problem 2.1 from the text.
   Find the set of solutions to the linear simultaneous equations $Ax = b$, with $A \in \mathbb{R}^{1 \times 1}$,
   $x \in \mathbb{R}^1$, and $b \in \mathbb{R}^1$, and where:
   
   (a) $A = [1]$, $b = [1]$,  
   (b) $A = [0]$, $b = [1]$,  
   (c) $A = [0]$, $b = [0]$.

2. Problem 2.2 from the text.
   Prove that $x^* = -3, 1$ are the only solutions to $g(x) = 0$, where $g : \mathbb{R} \rightarrow \mathbb{R}$ is given by
   
   $$\forall x \in \mathbb{R}, g(x) = (x)^2 + 2x - 3.$$
   
   (Hint: You must prove that any other value cannot satisfy the equation. A sketch of
   $g(x)$ versus $x$ may be useful to suggest an approach.)

3. Problem 2.3 from the text.
   Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by
   
   $$\forall x \in \mathbb{R}, f(x) = (x - 2)^2 + 1,$$
   
   and let $S = \mathbb{R}$.
   
   (a) Find $\min_{x \in S} f(x)$,
   (b) Find $\arg \min_{x \in S} f(x)$.

4. Problem 2.4 from the text.
   Show that any number $f \in \mathbb{R}$ such that $f \leq 1$ is a lower bound for the problem
   $\min_{x \in S} f(x)$, where $f : \mathbb{R} \rightarrow \mathbb{R}$ is the function defined as
   
   $$\forall x \in \mathbb{R}, f(x) = (x - 2)^2 + 1.$$
5. Problem 2.5 from the text.

Let $S \subseteq \mathbb{R}^n$ and $f : S \rightarrow \mathbb{R}$ and suppose that $f^*$ is the minimum of $\min_{x \in S} f(x)$. Also suppose that $\underline{f} \in \mathbb{R}$ satisfies $\underline{f} \leq f^*$. Show that $\underline{f}$ is a lower bound for $\min_{x \in S} f(x)$.

6. Problem 2.6 from the text.

Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by:

$$\forall x \in \mathbb{R}^2, h(x) = -x,$$

(that is, $\forall x \in \mathbb{R}^2$, $h_1(x) = -x_1$, $h_2(x) = -x_2$) and consider the constraints $h(x) \leq 0$. For each of the points

$$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x^{**} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x^{***} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

answer the following:

(a) Is $h_1(x) \leq 0$ active for the point?
(b) Is $h_2(x) \leq 0$ active for the point?
(c) What is the active set for the point?
(d) Is the point strictly feasible for the constraint $h_1(x) \leq 0$?
(e) Is the point strictly feasible for the constraint $h_2(x) \leq 0$?
(f) Is the point strictly feasible for the constraint $h(x) \leq 0$?
(g) Is the point on the boundary of $\{ x \in \mathbb{R}^2 | h(x) \leq 0 \}$?

Arrange your answer as a table with a column for each of the points $x^*$, $x^{**}$, and $x^{***}$ and seven rows for the seven parts of the question.

7. Problem 2.7 from the text.

Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by:

$$\forall x \in \mathbb{R}, f(x) = (x_1)^2 + (x_2)^2 + 2x_2 - 3.$$

(a) Sketch $C_f(\bar{f})$ for $\bar{f} = 0, 1, 2, 3$.
(b) Sketch on the same graph the set of points satisfying $g(x) = 0$ where $g : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$\forall x \in \mathbb{R}^2, g(x) = x_1 + 2x_2 - 3.$$

(c) Use your sketch to find the minimum and minimizer of

$$\min_{x \in \mathbb{R}^2} \{ f(x) | g(x) = 0 \}.$$
8. Problem 2.8 from the text.

Suppose that you own a data communications network that has just three customers: customers $k = 1, 2, 3$. The network consists of two backbone “links”: link $a$ that joins the point $X$ to point $Y$ and link $b$ that joins point $Y$ to $Z$, respectively. Links $a$ and $b$ have capacities $c_a$ and $c_b$, respectively, that represent the maximum bandwidth that the links can carry.

The three customers desire the following services.

- Customer 1 desires service from point $X$ to point $Y$, requiring bandwidth on link $a$.
- Customer 2 desires service from point $Y$ to point $Z$, requiring bandwidth on link $b$.
- Customer 3 desires service from point $X$ to point $Z$, requiring bandwidth on both link $a$ and link $b$.

Utilization of bandwidth on a link is additive. That is, the total load on a link is equal to the sum of the loads of the customers on that link. For each link, the total load on the link must be less than or equal to the link capacity of the load to be feasible.

You would like to allocate bandwidth to the customers in a systematic way. Somehow, the customers can communicate their “willingness-to-pay” or “utility” for services. That is, each customer $k$ can provide a function $f_k : \mathbb{R}_+ \to \mathbb{R}$ that represents how much customer $k$ values any particular desired (non-negative) level of service. Let us assume that these desires are commensurable; that is, they can legitimately be added together to determine overall utility of all the customers. (See Section 2.3.1.1 of the text.) We would like to maximize the overall value of all customers’ use of the network.

Cast this problem into an optimization problem of the form:

$$\max_{x \in \mathbb{R}^n} \{ f(x) | Ax = b, Cx \leq d \},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a function, $A$ and $C$ are matrices, and $b$ and $d$ are vectors. Make sure that the feasible set does not contain any points for which the functions $f_k$ are not defined. Specify the following explicitly:

(a) The variables in the formulation and their definitions. (Hint: Use $n = 3$ variables to represent the service delivered to the three customers.)

(b) The objective. (You can specify this function in terms of functions already defined.)

(c) The matrix and vector specifying the equality constraints, if any.

(d) The matrix and vector specifying the inequality constraints, if any.