Name: ____________________________________________

Instructions:

The examination lasts for 75 minutes and is closed book, closed notes. No electronic devices are permitted, including but not limited to calculators, cellphones, and other handheld devices. (Any such items in the examination room must be off and put away, subject to a 20 point penalty for the first violation and a score of 0 on the exam for the second violation.)

A table of properties of the discrete-time Fourier transform and some other information is provided for your convenience. There are five problems on the exam.

Do all your work on the pages in this exam booklet. Do not unstaple these pages. Any unstapled or restapled pages will NOT be graded. There is an extra worksheet following each problem page, and attached at the back of the exam booklet is one more extra work page. You may write on the backs of the pages if you need to.

Show your work and clearly indicate your final answers. Neatness and organization in your work is important and will influence your grade.

Each problem is weighted toward the final total as shown below.

Grades

1. ____________________ (20 pts.)
2. ____________________ (20 pts.)
3. ____________________ (20 pts.)
4. ____________________ (20 pts.)
5. ____________________ (20 pts.)
Total ____________________ (100 pts.)
Some properties of the discrete-time Fourier transform

<table>
<thead>
<tr>
<th>Property</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>$a x[n] + b v[n] \leftrightarrow a X(\Omega) + b X(\Omega)$</td>
</tr>
<tr>
<td>Time shift</td>
<td>$x[n - q] \leftrightarrow X(\Omega) e^{-j\omega q}, \ q \ \text{any integer}$</td>
</tr>
<tr>
<td>Time reversal</td>
<td>$x[-n] \leftrightarrow X(-\Omega) = \overline{X(\Omega)}$</td>
</tr>
<tr>
<td>Multiplication by $n$</td>
<td>$n x[n] \leftrightarrow j \frac{d}{d\Omega} X(\Omega)$</td>
</tr>
<tr>
<td>Multiplication by sinusoids</td>
<td>$x[n] e^{j\Omega_0 n} \leftrightarrow X(\Omega - \Omega_0), \ \text{for } \Omega_0 \ \text{real}$</td>
</tr>
<tr>
<td>Convolution</td>
<td>$x[n] \ast v[n] \leftrightarrow X(\Omega) V(\Omega)$</td>
</tr>
<tr>
<td>Multiplication</td>
<td>$x[n] v[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega - \lambda) V(\lambda) d\lambda$</td>
</tr>
<tr>
<td>Parseval’s theorem</td>
<td>$\sum_{n=-\infty}^{\infty} x[n] v[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{X(\Omega)} V(\Omega) d\Omega$</td>
</tr>
<tr>
<td>Special case of Parseval’s theorem</td>
<td>$\sum_{n=-\infty}^{\infty} x^2[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi}</td>
</tr>
</tbody>
</table>

Some discrete-time Fourier transform pairs

<table>
<thead>
<tr>
<th>$x[n]$</th>
<th>$X(\Omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta[n]$</td>
<td>1</td>
</tr>
<tr>
<td>$u[n]$</td>
<td>$\frac{1}{1 - e^{-\jmath \Omega}} + \sum_{k=-\infty}^{\infty} \pi \delta(\Omega - 2\pi k)$</td>
</tr>
<tr>
<td>$a^n u[n]$ with $</td>
<td>a</td>
</tr>
</tbody>
</table>
The sinc function is defined by

\[ \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}. \]

The discrete-time Fourier transform of a sinc function

\[ x[n] = \frac{B}{\pi} \text{sinc} \left( \frac{B}{\pi} n \right) \]
yields a DTFT given by a pulse:

\[ X(\Omega) = \begin{cases} 
1, & |\Omega| \leq B \\
0, & B < |\Omega| \leq \pi 
\end{cases} \]

\[ \sum_{k=n_0}^{n_1} a^n = \frac{a^{n_0} - a^{n_1+1}}{1-a}, \quad a \neq 1 \]
\[ \sum_{k=n_0}^{\infty} a^n = \frac{a^{n_0}}{1-a}, \quad |a| < 1 \]

\[ \frac{1 - e^{-j2N\omega}}{1 - e^{-j2M\omega}} = \frac{e^{-jN\omega}(e^{jN\omega} - e^{-jN\omega})}{e^{-jM\omega}(e^{jM\omega} - e^{-jM\omega})} = e^{-j(N-M)\omega} \frac{\sin(N\omega)}{\sin(M\omega)} \]

Common trigonometric function values

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \cos(\theta) )</th>
<th>( \sin(\theta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \pi/6 )</td>
<td>( \sqrt{3}/2 )</td>
<td>1/2</td>
</tr>
<tr>
<td>( \pi/4 )</td>
<td>( \sqrt{2}/2 )</td>
<td>( \sqrt{2}/2 )</td>
</tr>
<tr>
<td>( \pi/3 )</td>
<td>1/2</td>
<td>( \sqrt{3}/2 )</td>
</tr>
</tbody>
</table>

DFT formulae:

\[ x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j\frac{2\pi k}{N} n} \quad X_k = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N} n} \]
1. [20 points total] Let

\[ x[n] = 3 \left( \frac{1}{3} \right)^n u[n] . \]

and suppose that

\[ y[n] = (n - 3)x[n - 3] . \]

Find \( Y(\Omega) \).
Extra worksheet for problem 1
2. [20 points total] You have a sequence of eight time values \{x[0], x[1], \ldots, x[7]\} whose DFT is given by

\[ X_k = \begin{cases} 
3, & k = 2, 6 \\
0, & k = 0, 1, 3, 4, 5, 7 
\end{cases} \]

Use this information to express \( x[n] \) as a sum of real cosine and sine functions (e.g. \( x[n] = a_0 + a_1 \cos(\Omega_1 n + \theta_1) + b_1 \sin(\Omega_2 n + \phi_1) + a_2 \cos(\Omega_2 n + \theta_2) + b_2 \sin(\Omega_2 n + \phi_2) \) etc.).
Extra worksheet for problem 2
3. [20 points] An LTI system

\[
x(t) \rightarrow H(\omega) \rightarrow y(t)
\]

has frequency response

\[
H(\omega) = 3 \left( \frac{\sqrt{3}}{2} - j\omega \right) \left( \frac{\sqrt{3}}{2} + j\omega \right).
\]

Find \( y(t) \) if

\[
x(t) = 2 + \frac{1}{3} \cos \left( \frac{1}{2} t - \frac{\pi}{4} \right) - 2 \sin \left( \frac{\sqrt{3}}{2} t \right).
\]
Extra worksheet for problem 3
4. [20 points] We filter $x(t)$ to produce $y(t)$ via a continuous-time LTI system

\[
\begin{array}{c}
\text{x(t)} \\
\downarrow \\
H(\omega) \\
\uparrow \\
y(t)
\end{array}
\]

with the filter’s frequency response given by

\[H(\omega)\]

If $x(t)$ is given by the Fourier series expression

\[x(t) = 3 + \sum_{k=1}^{\infty} \frac{1}{k^2} \cos (k\pi t),\]

what is $y(t)$?
Extra worksheet for problem 4
5. **[20 points]** A discrete-time LTI system

\[
x[n] \xrightarrow{h[n]} y[n]
\]

has unit-pulse response

\[
h[n] = \begin{cases} 
1 & n = 0, 1, \ldots, 5 \\
0 & \text{otherwise}
\end{cases}
\]

What is \(y[n]\) if

\[
x[n] = \cos \left( \frac{\pi}{2} n - \frac{\pi}{16} \right)
\]

for all \(n\)?
Extra worksheet for problem 5
EXTRA WORKSHEET (indicate problem number clearly)