Name: __________________________________________

Instructions:
The examination lasts for 75 minutes and is closed book, closed notes. No electronic devices are permitted, including but not limited to calculators, cellphones, and other handheld devices. (Any such items in the examination room must be off and put away, subject to a 20 point penalty for the first violation and a score of 0 on the exam for the second violation.)

A table of properties of the discrete-time Fourier transform and some other information is provided for your convenience. There are five problems on the exam.

Do all your work on the pages in this exam booklet. **Do not unstaple these pages.** Any unstapled or restapled pages will NOT be graded. There is an extra worksheet following each problem page, and attached at the back of the exam booklet is one more extra work page. You may write on the backs of the pages if you need to.

Show your work and clearly indicate your final answers. Neatness and organization in your work is important and will influence your grade.

Each problem is weighted toward the final total as shown below.

**Grades**

1. ________________ (20 pts.)
2. ________________ (20 pts.)
3. ________________ (20 pts.)
4. ________________ (20 pts.)
5. ________________ (20 pts.)

Total ________________ (100 pts.)
Some properties of the discrete-time Fourier transform

<table>
<thead>
<tr>
<th>Property</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>$a x[n] + b v[n] \longleftrightarrow a X(\Omega) + b X(\Omega)$</td>
</tr>
<tr>
<td>Time shift</td>
<td>$x[n - q] \longleftrightarrow X(\Omega)e^{-jq\Omega}$, $q$ any integer</td>
</tr>
<tr>
<td>Time reversal</td>
<td>$x[-n] \longleftrightarrow X(-\Omega) = \overline{X(\Omega)}$</td>
</tr>
<tr>
<td>Multiplication by $n$</td>
<td>$n x[n] \longleftrightarrow \frac{j}{2\pi} \cdot X(\Omega)$</td>
</tr>
<tr>
<td>Multiplication by sinusoids</td>
<td>$x[n]e^{j\Omega_0 n} \longleftrightarrow X(\Omega - \Omega_0)$, for $\Omega_0$ real</td>
</tr>
<tr>
<td>Convolution</td>
<td>$x[n] * v[n] \longleftrightarrow X(\Omega)V(\Omega)$</td>
</tr>
<tr>
<td>Multiplication</td>
<td>$x[n]v[n] \longleftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega - \lambda)V(\lambda)d\lambda$</td>
</tr>
<tr>
<td>Parseval’s theorem</td>
<td>$\sum_{n=\infty} x[n] v[n] \longleftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{X(\Omega)}V(\Omega)d\Omega$</td>
</tr>
<tr>
<td>Special case of Parseval’s theorem</td>
<td>$\sum_{n=\infty} x^2[n] \longleftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi}</td>
</tr>
</tbody>
</table>

Some discrete-time Fourier transform pairs

<table>
<thead>
<tr>
<th>$x[n]$</th>
<th>$X(\Omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta[n]$</td>
<td>1</td>
</tr>
<tr>
<td>$u[n]$</td>
<td>$\frac{1}{1 - e^{-\pi}} + \sum_{k=-\infty}^{\infty} \pi \delta(\Omega - 2\pi k)$</td>
</tr>
<tr>
<td>$a^n u[n]$ with $</td>
<td>a</td>
</tr>
</tbody>
</table>
The sinc function is defined by

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}.$$  

The discrete-time Fourier transform of a sinc function

$$x[n] = \frac{B}{\pi} \text{sinc} \left( \frac{B}{\pi} n \right)$$

yields a DTFT given by a pulse:

$$X(\Omega) = \begin{cases} 1, & |\Omega| \leq B \\ 0, & B < |\Omega| \leq \pi \end{cases}$$

$$\sum_{k=n_0}^{n_1} a^n = \frac{a^{n_0} - a^{n_1+1}}{1 - a}, \quad a \neq 1$$

$$\sum_{k=n_0}^{\infty} a^n = \frac{a^{n_0}}{1 - a}, \quad |a| < 1$$

$$\frac{1 - e^{-j2N\omega}}{1 - e^{-j2M\omega}} = \frac{e^{-jN\omega}(e^{jN\omega} - e^{-jN\omega})}{e^{-jM\omega}(e^{jM\omega} - e^{-jM\omega})} = e^{-j(N-M)\omega} \frac{\sin(N\omega)}{\sin(M\omega)}$$

Common trigonometric function values

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\cos(\theta)$</th>
<th>$\sin(\theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\pi/6$</td>
<td>$\sqrt{3}/2$</td>
<td>1/2</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>$\sqrt{2}/2$</td>
<td>$\sqrt{2}/2$</td>
</tr>
<tr>
<td>$\pi/3$</td>
<td>1/2</td>
<td>$\sqrt{3}/2$</td>
</tr>
</tbody>
</table>
1. **[20 points total]** Let \( x[n] \leftrightarrow X(\Omega) \) be a DTFT pair, and let
\[
y[n] = x[n] + \frac{1}{2} x[-(n + 1)].
\]

(a) **[10 points]** Find \( Y(\Omega) \) in terms of \( X(\Omega) \).

(b) **[10 points]** Suppose that \( x[n] = \left(\frac{1}{2}\right)^{n} u[n] \).

For this \( x[n] \), find \( Y(\Omega) \) and express it as a single ratio (i.e. not as a sum of ratios). Also state whether \( Y(\Omega) \) is an even or odd function of \( \Omega \), or neither; and state whether \( Y(\Omega) \) is a purely real or purely imaginary function of \( \Omega \), or neither.
Extra worksheet for problem 1
2. [20 points total] Suppose that
\[ x[n] = c_0 + c_1 e^{j\frac{\pi}{2}n} + c_2 e^{j\pi n} + c_3 e^{j\frac{3\pi}{2}n} \]
for some (possibly complex) constants \( c_0, c_1, c_2, \) and \( c_3. \)

(a) [8 points] Show that \( x[n] \) is a periodic function of \( n \) with period \( N = 4. \)

(b) [8 points] Find
\[ \sum_{k=0}^{3} x[n] e^{-j\pi n} \]
in terms of \( c_0, c_1, c_2, \) and \( c_3. \) Simplify your expression as much as possible.

(c) [4 points] With \( \{X_0, X_1, X_2, X_3\} \) denoting the DFT of \( \{x[0], x[1], x[2], x[3]\}, \) find \( X_2 \) and find \( X_3. \)
Extra worksheet for problem 2
3. **[20 points]** A signal $x[n]$ is transmitted over a communications channel described by an LTI system with pulse response

$$h[n] = \frac{1}{2} \left( \frac{1}{4} \right)^n u[n].$$

You receive the channel’s output signal $y[n]$ as

$$y[n] = 2 \left( \frac{1}{2} \right)^n u[n] - \left( \frac{1}{4} \right)^n u[n].$$

What is the signal $x[n]$ that was transmitted?
Extra worksheet for problem 3
4. [20 points] You measure a combination of three signals

\[ x(t) = x_1(t) + x_2(t) + x_3(t) \]

with

\[ x_1(t) = 4 \]
\[ x_2(t) = -2 \cos \left( 2t + \frac{\pi}{4} \right) \]
\[ x_3(t) = 3 \cos \left( 3t - \frac{\pi}{2} \right) \]

You pass the combined signal \( x(t) \) through two different LTI systems

\[
\begin{array}{ccc}
  x(t) & \xrightarrow{H_A(\omega)} & y_A(t) \\
  & \downarrow & \\
  & \xrightarrow{H_B(\omega)} & y_B(t)
\end{array}
\]

where

\[ H_A(\omega) = \frac{\frac{4}{3}\omega(9 - \omega^2)}{(1 + j\frac{\omega}{2})^2}, \quad H_B(\omega) = \frac{\frac{1}{3}(\omega^2 - 4)}{(1 + j\frac{\omega}{3})^2}. \]

Find \( y_A(t) \) and \( y_B(t) \).
Extra worksheet for problem 4
5. [20 points] We filter $x(t)$ to produce $y(t)$ via a continuous-time LTI system

$$\begin{array}{c}
x(t) \rightarrow H(\omega) \rightarrow y(t)
\end{array}$$

with the filter’s frequency response given by

$$H(\omega)$$

If $x(t)$ is given by the Fourier series expression

$$x(t) = 6 + \sum_{k=1}^{\infty} \frac{10}{(2k+1)(2k-1)} \cos((2k-1)2t),$$

what is $y(t)$?
Extra worksheet for problem 5
EXTRA WORKSHEET (indicate problem number clearly)