Name: _______________________________________

Instructions:
The examination lasts for 75 minutes and is closed book, closed notes. No electronic devices are permitted, including but not limited to calculators, cellphones, and other handheld devices. (Any such items in the examination room must be off and put away, subject to a 20 point penalty for the first violation and a score of 0 on the exam for the second violation.)

Tables of properties of the Fourier transform and the discrete-time Fourier transform are attached for your convenience, as is a brief table of Fourier transform pairs, and also a variety of mathematical formulae. Theses may or may not be needed to solve the exam problems.

Do all your work on the pages in this exam booklet. Do not unstaple these pages. Any unstapled or restapled pages will NOT be graded. You may write on the backs of the pages if you need to, and attached at the back of the exam booklet are two extra work pages.

Show your work and clearly indicate your final answers. Neatness and organization in your work is important and will influence your grade.

Each problem is weighted toward the final total as shown below.

Grades

1. ___________________ (15 pts.)
2. ___________________ (30 pts.)
3. ___________________ (20 pts.)
4. ___________________ (15 pts.)
5. ___________________ (10 pts.)
6. ___________________ (10 pts.)
Total ___________________ (100 pts.)
### Some properties of the Fourier transform

<table>
<thead>
<tr>
<th>Property</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Linearity</strong></td>
<td>$ax(t) + bv(t) \longleftrightarrow aX(\omega) + bX(\omega)$</td>
</tr>
<tr>
<td><strong>Time shift</strong></td>
<td>$x(t - c) \longleftrightarrow X(\omega)e^{-j\omega c}$</td>
</tr>
<tr>
<td><strong>Time scaling</strong></td>
<td>$x(at) \longleftrightarrow \frac{1}{a}X\left(\frac{\omega}{a}\right)$, for $a &gt; 0$</td>
</tr>
<tr>
<td><strong>Time reversal</strong></td>
<td>$x(-t) \longleftrightarrow X(-\omega)$</td>
</tr>
<tr>
<td><strong>Multiplication by a power of $t$</strong></td>
<td>$t^n x(t) \longleftrightarrow j^n \frac{d^n}{d\omega^n}X(\omega)$, $n = 1, 2, \ldots$</td>
</tr>
<tr>
<td><strong>Multiplication by sinusoids</strong></td>
<td>$e^{j\omega_0 t} x(t) \longleftrightarrow X(\omega - \omega_0)$, for $\omega_0$ real</td>
</tr>
<tr>
<td><strong>Differentiation</strong></td>
<td>$\frac{d^n}{dt^n} x(t) \longleftrightarrow (j\omega)^n X(\omega)$, $n = 1, 2, \ldots$</td>
</tr>
<tr>
<td><strong>Integration</strong></td>
<td>$\int_{-\infty}^{t} x(\lambda)d\lambda \longleftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$</td>
</tr>
<tr>
<td><strong>Convolution</strong></td>
<td>$x(t) * v(t) \longleftrightarrow X(\omega)V(\omega)$</td>
</tr>
<tr>
<td><strong>Multiplication</strong></td>
<td>$x(t)v(t) \longleftrightarrow \frac{1}{2\pi} X(\omega) * V(\omega)$</td>
</tr>
<tr>
<td><strong>Duality</strong></td>
<td>$X(t) \longleftrightarrow 2\pi x(-\omega)$</td>
</tr>
</tbody>
</table>
### Some properties of the discrete-time Fourier transform

<table>
<thead>
<tr>
<th>Property</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>( ax[n] + bv[n] \leftrightarrow aX(\Omega) + bX(\Omega) )</td>
</tr>
<tr>
<td>Time shift</td>
<td>( x[n - q] \leftrightarrow X(\Omega)e^{-jq\Omega}, \ q \ \text{any integer} )</td>
</tr>
<tr>
<td>Time reversal</td>
<td>( x[-n] \leftrightarrow X(-\Omega) = \overline{X(\Omega)} )</td>
</tr>
<tr>
<td>Multiplication by ( n )</td>
<td>( nx[n] \leftrightarrow j\frac{d}{d\Omega}X(\Omega) )</td>
</tr>
<tr>
<td>Multiplication by sinusoids</td>
<td>( x[n]e^{j\Omega_0n} \leftrightarrow X(\Omega - \Omega_0), \ \text{for} \ \Omega_0 \ \text{real} )</td>
</tr>
<tr>
<td>Convolution</td>
<td>( x[n] \ast v[n] \leftrightarrow X(\Omega)V(\Omega) )</td>
</tr>
<tr>
<td>Multiplication</td>
<td>( x[n]v[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega - \lambda)V(\lambda)d\lambda )</td>
</tr>
<tr>
<td>Parseval’s theorem</td>
<td>( \sum_{n=-\infty}^{\infty} x[n]v[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{X(\Omega)}V(\Omega)d\Omega )</td>
</tr>
<tr>
<td>Special case of Parseval’s theorem</td>
<td>( \sum_{n=-\infty}^{\infty} x^2[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi}</td>
</tr>
</tbody>
</table>
Some Fourier transform pairs

\[
\begin{align*}
\delta(t) & \iff 1 \\
u(t) & \iff \frac{1}{j\omega} + \pi \delta(0) \\
1 & \iff 2\pi \delta(\omega) \\
e^{j\omega_0 t} & \iff 2\pi \delta(\omega - \omega_0) \\
e^{-at}u(t), \ a > 0 & \iff \frac{1}{j\omega + a}
\end{align*}
\]

Some trigonometric identities

\[
\begin{align*}
sin(\alpha \pm \beta) &= sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta) \\
cos(\alpha \pm \beta) &= cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta) \\
\sin(\alpha) \cos(\beta) &= \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)] \\
cos(\alpha) \cos(\beta) &= \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)] \\
\sin(\alpha) \sin(\beta) &= \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]
\end{align*}
\]

The sinc function is defined by

\[
sinc(x) = \frac{\sin(\pi x)}{\pi x}.
\]

The discrete-time Fourier transform of a sinc function time signal

\[
x[n] = \frac{B}{\pi} \text{sinc} \left( \frac{B}{\pi} n \right)
\]

yields a DTFT given by a pulse

\[
X(\Omega) = \begin{cases} 
1, & |\Omega| \leq B \\
0, & B < |\Omega| \leq \pi 
\end{cases}
\]

\[
\sum_{k=n_0}^{n_1} a^n = \frac{a^{n_0} - a^{n_1+1}}{1 - a}, \ a \neq 1 \\
\sum_{k=n_0}^{\infty} a^n = \frac{a^{n_0}}{1 - a}, \ |a| < 1
\]

\[
\frac{1 - e^{-j2N\omega}}{1 - e^{-j2M\omega}} = e^{-jN\omega} \left( e^{jN\omega} - e^{-jN\omega} \right) e^{-jM\omega} \left( e^{jM\omega} - e^{-jM\omega} \right) = e^{-j(N-M)\omega} \frac{\sin(N\omega)}{\sin(M\omega)}
\]
1. [15 points] The signal $x(t)$ shown below has a Fourier series expansion given by

$$x(t) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{-2}{(k\pi)^2} (1 - (-1)^k) \cos(k\pi t) + \sum_{k=0}^{\infty} \frac{2}{k\pi} (1 - (-1)^k) \sin(k\pi t).$$

Note that $x(t) = x_1(t) + x_2(t)$

where $x_1(t)$ and $x_2(t)$ are given by the following plots.

Find Fourier series expansions for $x_1(t)$ and $x_2(t)$. 


EXTRA WORKSHEET for problem 1
2. **[30 points total]** The signum function $\text{sgn}(t)$ is defined by

$$\text{sgn}(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$

(a) **[15 points]** Show that the Fourier transform of $\text{sgn}(t)$ is given by

$$\text{SGN}(\omega) = \frac{2}{j\omega}.$$  

[Hint: can you express $\text{sgn}(t)$ in terms of a step function, or step functions, possibly reversed in time?]

(b) **[15 points]** Find the Fourier transform of

$$h(t) = \frac{1}{t}, \quad -\infty < t < \infty$$
3. [20 points] Suppose $h(t)$ is a signal whose Fourier transform is

$$H(\omega) = \begin{cases} -j, & \omega > 0 \\ j, & \omega < 0 \end{cases}$$

and let $y(t) = h(t) * x(t)$. Find $Y(\omega)$ and $y(t)$ when $x(t) = A \cos(\omega_0 t)$. 
4. [15 points] Find the discrete-time Fourier transform $X(\Omega)$ for

$$x[n] = n \left( \frac{1}{2} \right)^n u[n].$$
5. [10 points] You know that in the interval $-\pi < \Omega < \pi$, the discrete-time Fourier transform $X(\Omega)$ of a signal $x[n]$ is given by

$$
\frac{6}{3 + |\Omega|}.
$$

Determine the value of $X(4\pi)$.

6. [10 points] A signal $x[n]$ has a purely real-valued discrete-time Fourier transform $X(\Omega)$ given by

$$
X(\Omega) = \begin{cases} 
6, & |\Omega| \leq \pi/4 \\
0, & \pi/4 < |\Omega| \leq \pi
\end{cases}
$$

Is the time signal $x[n]$ an even function of $n$, an odd function of $n$, or neither?
EXTRA WORKSHEET (indicate problem number clearly)
EXTRA WORKSHEET (indicate problem number clearly)