Name: __________________________________________________________

Instructions:

The examination lasts for 75 minutes and is closed book, closed notes. No calculators permitted. A table of properties of the discrete-time Fourier transform and some other information is attached for your convenience. There are five problems on the exam.

Do all your work on the pages in this exam booklet. **Do not unstaple these pages. Any unstapled or restapled pages will NOT be graded.** There is an extra worksheet following each problem page, and attached at the back of the exam booklet is one more extra work page.

Show your work and clearly indicate your final answers. Neatness and organization in your work is important and will influence your grade.

Each problem is weighted toward the final total as shown below.

**Grades**

1. __________________ (20 pts.)
2. __________________ (20 pts.)
3. __________________ (20 pts.)
4. __________________ (20 pts.)
5. __________________ (20 pts.)

Total _______________ (100 pts.)
Some properties of the discrete-time Fourier transform

<table>
<thead>
<tr>
<th>Property</th>
<th>Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>$ax[n] + bv[n] \leftrightarrow aX(\Omega) + bX(\Omega)$</td>
</tr>
<tr>
<td>Time shift</td>
<td>$x[n - q] \leftrightarrow X(\Omega)e^{-jq\Omega}$, $q$ any integer</td>
</tr>
<tr>
<td>Time reversal</td>
<td>$x[-n] \leftrightarrow X(-\Omega) = \overline{X(\Omega)}$</td>
</tr>
<tr>
<td>Multiplication by $n$</td>
<td>$nx[n] \leftrightarrow j\frac{d}{d\Omega}X(\Omega)$</td>
</tr>
<tr>
<td>Multiplication by sinusoids</td>
<td>$x[n]e^{j\Omega_0 n} \leftrightarrow X(\Omega - \Omega_0)$, for $\Omega_0$ real</td>
</tr>
<tr>
<td>Convolution</td>
<td>$x[n] * v[n] \leftrightarrow X(\Omega)V(\Omega)$</td>
</tr>
<tr>
<td>Multiplication</td>
<td>$x[n]v[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega - \lambda)V(\lambda)d\lambda$</td>
</tr>
<tr>
<td>Parseval’s theorem</td>
<td>$\sum_{n=-\infty}^{\infty} x[n]v[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{X(\Omega)V(\Omega)}d\Omega$</td>
</tr>
<tr>
<td>Special case of Parseval’s theorem</td>
<td>$\sum_{n=-\infty}^{\infty} x^2[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi}</td>
</tr>
</tbody>
</table>

Some discrete-time Fourier transform pairs

<table>
<thead>
<tr>
<th>$x[n]$</th>
<th>$X(\Omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta[n]$</td>
<td>1</td>
</tr>
<tr>
<td>$u[n]$</td>
<td>$\frac{1}{1 - e^{-\pi}} + \sum_{k=-\infty}^{\infty} \pi \delta(\Omega - 2\pi k)$</td>
</tr>
<tr>
<td>$a^n u[n]$ with $</td>
<td>a</td>
</tr>
</tbody>
</table>
The sinc function is defined by
\[ \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}. \]

The discrete-time Fourier transform of a sinc function

\[ x[n] = \frac{B}{\pi} \text{sinc} \left( \frac{B}{\pi} n \right) \]

yields a DTFT given by a pulse:

\[ X(\Omega) = \begin{cases} 1, & \text{if } |\Omega| \leq B \\ 0, & \text{if } B < |\Omega| \leq \pi \end{cases} \]

\[
\sum_{k=n_0}^{n_1} a^n = \frac{a^{n_0} - a^{n_1+1}}{1 - a}, \quad a \neq 1
\]

\[
\sum_{k=n_0}^{\infty} a^n = \frac{a^{n_0}}{1 - a}, \quad |a| < 1
\]

\[
1 - e^{-j2N\omega} = \frac{e^{-jN\omega}(e^{jN\omega} - e^{-jN\omega})}{e^{-jM\omega}(e^{jM\omega} - e^{-jM\omega})} = e^{-j(N-M)\omega} \frac{\sin(N\omega)}{\sin(M\omega)}
\]

<table>
<thead>
<tr>
<th>Common trigonometric function values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta )</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>( \frac{\pi}{6} )</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
</tr>
<tr>
<td>( \frac{\pi}{3} )</td>
</tr>
</tbody>
</table>
1. [20 points total] We know that

\[ x[n] = \left( \frac{1}{2} \right)^n u[n] \leftrightarrow \frac{1}{1 - \frac{1}{2} e^{-j\Omega}} = X(\Omega) \]

form a DTFT pair.

(a) [10 points] Find the DTFT of

\[ y[n] = (n - 1) \left( \frac{1}{2} \right)^{n-1} u[n - 1]. \]

(b) [10 points] What is \( y[n] \) if

\[ Y(\Omega) = \frac{1/2}{1 - \frac{1}{2} e^{-j(\Omega - \pi)}} + \frac{1/2}{1 - \frac{1}{2} e^{-j(\Omega + \pi)}}. \]
Extra worksheet for problem 1
2. [20 points total] Recall that the DFT is given by

\[ X_k = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi n k}{N}}, \quad k = 0, 1, \ldots, N - 1 \]

and that the inverse DFT is given by

\[ x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j\frac{2\pi k n}{N}}, \quad n = 0, 1, \ldots, N - 1. \]

Suppose that \( N = 12 \) and

\[ x[n] = e^{j\frac{5\pi}{6} n} = e^{j\frac{2\pi 5}{12} n} \]

(a) [10 points] What is \( X_5 \)?

(b) [10 points] What is \( X_k \) for \( k = 0, 1, \ldots, 4 \) and \( k = 6, 7, \ldots, 11 \)?
Extra worksheet for problem 2
3. [20 points total] When

\[ x[n] = \left( \frac{1}{2} \right)^n u[n] \]

is the input to an LTI system, the output is

\[ y[n] = 2 \left( \frac{1}{2} \right)^n u[n] - \left( \frac{1}{4} \right)^n u[n]. \]

(a) [10 points] Find the system’s frequency response \( H(\Omega) \).

(b) [10 points] What is the output \( y[n] \) if instead the input were

\[ x[n] = \left( \frac{1}{5} \right)^n u[n]? \]
Extra worksheet for problem 3
4. [20 points] A continuous-time LTI system

\[
x(t) \xrightarrow{X(\omega)} H(\omega) \xrightarrow{Y(\omega)} y(t)
\]

has frequency response

\[
H(\omega) = \frac{9 - \omega^2}{(2 + j\omega)^2}.
\]

Find \(y(t)\) if

\[
x(t) = 3 + 2 \cos(2t + \pi/4) - 3 \cos(3t).
\]
Extra worksheet for problem 4
5. **[20 points]** We filter $x[n]$ to produce $y[n]$ via a discrete-time LTI system

$$
\begin{array}{c}
x[n] \\
H(\Omega) \\
y[n]
\end{array}
$$

with the filter’s frequency response given by

$$H(\Omega) = \begin{cases} 
1 & \text{if } -\pi < \Omega < \pi \\
0 & \text{otherwise}
\end{cases}$$

If

$$x[n] = \sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^k \cos \left( \frac{\pi}{2k} n \right),$$

what is $y[n]$?
Extra worksheet for problem 5
EXTRA WORKSHEET (indicate problem number clearly)